

A pentagon of identities, graded tensor products, and the Kirillov-Reshetikhin conjecture

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This paper provides a brief review of the relations between the Feigin-Loktev conjecture on the dimension of graded tensor products of $\mathfrak{gl}[t]$ -modules, the Kirillov-Reshetikhin conjecture, the combinatorial “ $M = N$ ” conjecture, their proofs for all simple Lie algebras, and a pentagon of identities which results from the proof.

Keywords: Fusion products; KR modules

Dedicated to T. Miwa on the occasion of his 60th birthday

1. Introduction

This paper reviews work which followed^{1–3} the author’s fruitful collaboration with T. Miwa and colleagues.^{4,5} This work was inspired by the work of Feigin and Loktev on fusion products.⁶ The series of results described here finally culminated in a proof^{2,3} of the Feigin-Loktev conjecture concerning the graded character of the (non-level restricted) fusion product, in the case of special modules known as Kirillov-Reshetikhin modules. The purpose of this article is to make clear the sequence of connections and relations between the various results which lead to the proof.

The fusion product character first appeared in the ‘80’s, in the work on the completeness conjecture for Bethe Ansatz states⁷ for the generalized Heisenberg models. The completeness conjecture is one version of what later became known as the Kirillov-Reshetikhin conjecture, and involves the first appearance of an object called the generalized Kostka polynomial. The (generalized) Kostka number gives the decomposition coefficients of tensor products of KR-modules into irreducible components. Although much work was later published on the subject, the conjecture in its original, combina-

torial form – counting solutions of the Bethe equations – was only proved to be true in special cases. In other cases, a similar but not manifestly positive formula^{8–10} was shown to hold.

The key to proving both the Kirillov-Reshetikhin conjecture and the Feigin-Loktev conjecture is a combinatorial identity, the equality of two polynomials in q , one written as an alternating sum, and the other as a sum of positive terms. The deeper meaning of this identity remains mysterious, but its proof³ using purely combinatorial means finally implies several equalities, proving the conjectures above for any simple Lie algebras.

1.1. *The objects of interest*

We describe several objects and relations between their dimensions. (Section 2 contains a fuller discussion of several of these as necessary).

1.1.1. *Kirillov-Reshetikhin modules*

These are finite-dimensional modules of the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ or the Yangian $Y(\mathfrak{g})$. Let \mathfrak{g} be a simple Lie algebra of rank r with Cartan matrix C . Consider the irreducible, finite-dimensional \mathfrak{g} -module V with a highest weight which is a non-negative multiple of a fundamental weight. The Yangian $Y(\mathfrak{g})$ contains \mathfrak{g} as a subalgebra. The irreducible $Y(\mathfrak{g})$ -module induced from V is called a Kirillov-Reshetikhin module.⁷ It is finite-dimensional, and its \mathfrak{g} -highest weight is that of V . In the case where $\mathfrak{g} = A_n$, it is equal to V as a \mathfrak{g} -module. In other cases, the restriction to a \mathfrak{g} -module may or may not be irreducible, but in that case, V is always a component in the decomposition, with multiplicity 1, and with the highest weight.

Equivalently, one may consider Kirillov-Reshetikhin modules for the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ and their similar decomposition into $U_q(\mathfrak{g})$ -modules.¹¹ These are also referred to as KR-modules.

We denote the KR-modules by $KR_{\alpha,m}(\zeta)$, where $1 \leq \alpha \leq r$ and m is a positive integer. These have a \mathfrak{g} -highest weight of the form $m\omega_\alpha$, where ω_α is a fundamental weight of \mathfrak{g} . The parameter ζ is a complex number which is called the spectral parameter.

1.1.2. *Chari's graded $\mathfrak{g}[t]$ -modules*

These are modules of the current algebra $\mathfrak{g}[t]$, defined as a quotient of $U(\mathfrak{n}_-[t])$ by an ideal generated by relations¹¹ (see Equations (12),(11)).

The relations are the $q \rightarrow 1$ limits of the similar relations which hold in the quantum case for Kirillov-Reshetikhin modules. These modules also have a \mathfrak{g} -highest weight equal to a multiple of one of the fundamental weights of \mathfrak{g} , as in the quantum algebra case. We denote this module by $C_{\alpha,m}(\zeta)$, where the highest weight is again $m\omega_\alpha$.

1.1.3. Decomposition of tensor products

We observe that by general deformation arguments, the dimension of KR-modules is bounded from above by that of Chari's modules.

More precisely, the decomposition coefficients of the KR-modules, and therefore their tensor products at generic values of the spectral parameters, into irreducible \mathfrak{g} -modules are bounded from above by those of Chari's modules. That is, choose a sequence of non-negative integers $\mathbf{n} = \{n_m^{(\alpha)} : 1 \leq \alpha \leq r, m > 0\}$ and consider the multiplicities $M_{\lambda,\mathbf{n}}^Y$ and $M_{\lambda,\mathbf{n}}^{\mathfrak{g}[t]}$, defined by

$$M_{\lambda,\mathbf{n}}^Y = \dim \operatorname{Hom}_{\mathfrak{g}} \left(\bigotimes_{\alpha,m} \operatorname{KR}_{\alpha,m}^{\otimes n_m^{(\alpha)}}, V(\lambda) \right),$$

where $V(\lambda)$ is the irreducible highest weight \mathfrak{g} -module with highest weight λ , and

$$M_{\lambda,\mathbf{n}}^{\mathfrak{g}[t]} = \dim \operatorname{Hom}_{\mathfrak{g}} \left(\bigotimes_{\alpha,m} C_{\alpha,m}^{\otimes n_m^{(\alpha)}}, V(\lambda) \right).$$

(Here, we omitted the dependence of the modules on the spectral parameter: We assume that all spectral parameters are taken at generic values with respect to each other). Then we have the first inequality:

$$M_{\lambda,\mathbf{n}}^Y \leq M_{\lambda,\mathbf{n}}^{\mathfrak{g}[t]}, \quad (1)$$

which simply follows by general deformation arguments: Both are defined as quotients by some ideal, and the ideal in the limit $q \rightarrow 1$ may be smaller than that for generic values of q .

1.1.4. The combinatorial KR-conjecture: The M -sum formula

This is a conjecture that $M_{\lambda,\mathbf{n}}^Y$ is equal to the number of Bethe vectors. The generalized, inhomogeneous Heisenberg spin chain has a Hilbert space which is equal, by definition, to the tensor product of Yangian modules,

$$\mathcal{H}_{\mathbf{n}} = \prod_{\alpha,m} \operatorname{KR}_{\alpha,m}^{\otimes n_m^{(\alpha)}}.$$

Again, the modules are taken at generic values of the spectral parameters, that is, pairwise not separated by an integer. (Note that the model is also well-defined for any other finite-dimensional $Y(\mathfrak{g})$ -modules, but no Bethe Ansatz solution is known generically.)

This model has a Bethe Ansatz solution. The completeness conjecture of Kirillov and Reshetikhin⁷ states that there is a Bethe vector for each \mathfrak{g} -highest weight vector in $\mathcal{H}_{\mathbf{n}}$. In particular, there is an explicit formula for the number of Bethe vectors, and in fact, the authors wrote down a graded formula (which we now know has a direct interpretation as a grading by the linearized energy function of the model), although at the time, the meaning of the refinement was unknown. For the \mathfrak{g} -highest weight λ , the multiplicity is the number $M_{\lambda, \mathbf{n}}$ obtained as the $q \rightarrow 1$ limit of the following, grading-endowed formula:⁷

$$M_{\lambda, \mathbf{n}}(q) = \sum_{\mathbf{m}} q^{Q(\mathbf{m}, \mathbf{n})} \prod_{\alpha, i} \begin{bmatrix} m_i^{(\alpha)} + P_i^{(\alpha)} \\ m_i^{(\alpha)} \end{bmatrix}_q \quad (2)$$

where the sum is taken over the non-negative integers $\mathbf{m} = \{m_i^{(\alpha)} : 1 \leq \alpha \leq r, i \geq 1\}$ such that $\sum_i m_i^{(\alpha)} = m^{(\alpha)}$, where $m^{(\alpha)}$ are integers fixed by the “zero weight condition”

$$\sum_{\beta} C_{\alpha, \beta} m^{(\alpha)} = \sum_i n_i^{(\alpha)} - \ell^{(\alpha)}, \quad (3)$$

$\ell^{(\alpha)}$ being the coefficient of ω_{α} in the weight λ . Note that this sum has only a finite number of non-vanishing terms. Let us define

$$B_{i, j}^{(\alpha, \beta)} = \text{sign}(C_{\alpha, \beta}) \min(|C_{\alpha, \beta}|j, |C_{\beta, \alpha}|i)$$

Then the vacancy numbers $P_i^{(\alpha)}$ are defined as

$$P_i^{(\alpha)} = \sum_j \min(i, j) n_j^{(\alpha)} - (B\mathbf{m})_i^{(\alpha)}. \quad (4)$$

and the quadratic form $Q(\mathbf{m}, \mathbf{n})$ is

$$Q(\mathbf{m}, \mathbf{n}) = \frac{1}{2} \mathbf{m} \cdot \mathbf{P}. \quad (5)$$

The q -binomial coefficient is defined as usual, and in the limit $q \rightarrow 1$ becomes the usual binomial coefficient. In particular, *the sum is taken over the restricted set of integers \mathbf{m} such that $P_j^{(\alpha)} \geq 0$.*

This provides a formula for the tensor multiplicities $M_{\lambda, \mathbf{n}}^Y$. It was proved in several special cases using combinatorial means.^{7, 12–16} In general, a similar but not equivalent formula was known to be true, as explained below.

1.1.5. The HKOTY N -sum formula

For general Lie algebras, and for generic KR-modules, the following formula was conjectured:⁸

$$M_{\lambda, \mathbf{n}}^Y = \lim_{q \rightarrow 1} N_{\lambda, \mathbf{n}}(q), \quad (6)$$

where $N_{\lambda, \mathbf{n}}(q)$ is a modified form of the formula (2), obtained by simply removing the restriction $P_j^{(\alpha)} \geq 0$. Both the usual and the q -binomial coefficients are defined when $P_j^{(\alpha)} < 0$, but they carry a sign in that case. The authors conjectured, after extensive testing, that all terms coming from sets \mathbf{m} such that $P_j^{(\alpha)} < 0$ for some j, α cancel, so that

$$M_{\lambda, \mathbf{n}}(q) = N_{\lambda, \mathbf{n}}(q). \quad (7)$$

The conjecture (6) holds provided that the so-called Q -system⁷ is satisfied by the characters of KR-modules. It was shown by Nakajima (for simply-laced algebras)⁹ and Hernandez for all other Lie algebras¹⁷ that the q -characters of KR-modules satisfy the more general T -system,¹⁸ from which the Q -system follows. Hence, Equation (6) had achieved the status of a Theorem.

1.1.6. Feigin-Loktev fusion products

The Feigin-Loktev fusion product is a graded $\mathfrak{g}[t]$ -module,⁶ which is a refinement of the usual tensor product of \mathfrak{g} -module (cyclic, finite-dimensional). One chooses a finite-dimensional cyclic \mathfrak{g} -module V , from which one induces an action of the current algebra $\mathfrak{g}[t]$ localized at some complex number ζ . More specifically, one defines a graded tensor product by choosing N \mathfrak{g} -modules V_i , each with a cyclic vector v_i , localized at N distinct points in \mathbb{CP} . One then defines the fusion product as the associated graded space of the filtered space, generated by the action of $U(\mathfrak{g}[t])$ on the tensor product of cyclic vectors, with the grading defined by degree in t . This is a graded space, and the graded components are \mathfrak{g} -modules.

Feigin and Loktev conjectured that the fusion product as a graded space is independent of the localization parameters for sufficiently well-behaved \mathfrak{g} -modules. Moreover, they conjectured a relation between the graded coefficients of the \mathfrak{g} -module $V(\lambda)$ in the fusion product, and the generalized Kostka polynomials.¹⁹ This conjecture was proved for \mathfrak{sl}_2 in,⁵ and in greater generality in several other works.

In particular, in,² we proved the following inequality, using techniques generalized from.⁵ Let $\mathcal{F}_{\mathbf{n}}^*$ be the fusion product of the modules $C_{\alpha, m}$ with

multiplicity $n_m^{(\alpha)}$. This is a graded space. We define the q -dimension to be the Hilbert polynomial of the graded space. Then

$$q\text{-dim Hom}_{\mathfrak{g}}(\mathcal{F}_{\mathbf{n}}^*, V(\lambda)) \leq M_{\lambda, \mathbf{n}}(q), \quad (8)$$

where $M_{\lambda, \mathbf{n}}$ is the fermionic formula of Kirillov and Reshetikhin for the number of Bethe vectors in the generalized, inhomogeneous Heisenberg spin chain corresponding to KR-modules $KR_{\alpha, m}$ with the same multiplicities.

Remark 1.1. The inequality in (8) arises from the following sequence of maps: One may completely characterize the dual space of functions of the fusion product in terms of symmetric functions with certain zeros and poles (we do this in Section 3). Actual calculation of the Hilbert polynomial of this space requires another injective mapping into another filtered space, whose Hilbert polynomial is the polynomial $M_{\lambda, \mathbf{n}}(q)$. We do not prove the surjectivity of the map, resulting in the inequality in Equation (8).

Moreover, the space $\mathcal{F}_{\mathbf{n}}^*$ is the associated graded space of the tensor product of Chari modules, which are defined as a quotient of $U(\mathfrak{g}[t])$ by a certain ideal. Again, by a general deformation argument, we have that

$$\dim \text{Hom}_{\mathfrak{g}}(\otimes C_{\alpha, i}^{\otimes n_i^{(\alpha)}}, V(\lambda)) \leq \dim \text{Hom}_{\mathfrak{g}}(\mathcal{F}_{\mathbf{n}}^*, V(\lambda)).$$

Note that the sum on the right hand side of Equation (8) is manifestly positive, and therefore if, in the $q \rightarrow 1$ limit, it is equal to the tensor product multiplicity, then we have the equality of graded spaces also, since the left-hand side has a dimension which is greater than or equal to the tensor product multiplicity by the deformation argument.

1.2. A pentagon of identities

We have a sequence of identities and inequalities:

$$\begin{array}{ccc} |\text{Hom}_{\mathfrak{g}}(\mathcal{F}_{\mathbf{n}}^*, V(\lambda))| & & \\ \swarrow & & \searrow \\ \left| \text{Hom}_{\mathfrak{g}} \left(\otimes_{\alpha, m} C_{\alpha, m}^{\otimes n_m^{(\alpha)}} , V(\lambda) \right) \right| & & M_{\lambda, \mathbf{n}} \\ \vee & & \parallel \leftarrow \text{final step} \\ \left| \text{Hom}_{U_q(\mathfrak{g})} \left(\otimes_{\alpha, m} KR_{\alpha, m}^{\otimes n_m^{(\alpha)}} , V(\lambda) \right) \right| & = & N_{\lambda, \mathbf{n}} \end{array}$$

The “final step” remaining in this pentagon was to prove the conjectured identity (7). The proof turns all the inequalities in the pentagon

to equalities. This conjecture was proven by combinatorial means³ for all simple Lie algebras. Therefore, this provides a proof of the completeness problem in the Bethe Ansatz known as the Kirillov-Reshetikhin conjecture, as well as the Feigin-Loktev conjecture for the cases of Kirillov-Reshetikhin conjecture.

1.3. Plan of the paper

In the following sections, we will summarize the proof² of the inequality (8) and the proof of the $M = N$ conjecture.³ In Section 2, we give a definition of the Feigin-Loktev fusion product of Chari's modules. In Section 3, we summarize the proof of the inequality (8) which is obtained via a functional realization of the multiplicity space, following the ideas of B. Feigin. In Section 4, we explain the combinatorial proof of the $M = N$ conjecture.³

2. Definitions

Here, we add some details to the definitions of the representation-theoretical objects which are important in the theorems below.

Let \mathfrak{g} be a simple Lie algebra of rank r and Cartan matrix C . Let $\mathfrak{g}[t] = \mathfrak{g} \otimes \mathbb{C}[t]$ be the corresponding Lie algebra of polynomials in t with coefficients in \mathfrak{g} .

2.1. Finite-dimensional $\mathfrak{g}[t]$ -modules and the fusion action

Given a complex number ζ , any \mathfrak{g} -module V can be extended to a $\mathfrak{g}[t]$ -module evaluation module $V(\zeta)$, with t evaluated at ζ . The generators $x[n] := x \otimes t^n$ ($x \in \mathfrak{g}$) act on $v \in V$ as $\pi(x[n])v = \zeta^n xv$.

The dimension of the evaluation module is the same as that of V . If V is irreducible as a \mathfrak{g} -module, so is $V(\zeta)$.

More generally, given a $\mathfrak{g}[t]$ -module V , the $\mathfrak{g}[t]$ -module localized at ζ , $V(\zeta)$, is the module on which $\mathfrak{g}[t]$ acts by expansion in the local parameter $t_\zeta := t - \zeta$. If $v \in V(\zeta)$, then

$$x[n]v = x \otimes (t_\zeta + \zeta)^n v = \sum_j \binom{n}{j} \zeta^j x[n-j]_\zeta v,$$

where $x[n]_\zeta := x \otimes t_\zeta$ and $x[n]_\zeta$ acts on $v \in V(\zeta)$ in the same way that $x[n]$ acts on $v \in V$.

Another way to write this is in terms of generating functions, for any

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$x \in \mathfrak{g}$,

$$x(z) = \sum_{n \in \mathbb{Z}} x[n] z^{-n-1}.$$

Then if $\zeta \in \mathbb{C}$,

$$x[n]_{\zeta} = \frac{1}{2\pi i} \oint_{z=\zeta} (z - \zeta)^n x(z) dz. \quad (9)$$

We will also need to be able to localize modules at infinity. In that case,

$$x[n]_{\infty} = \frac{1}{2\pi i} \oint_{z=\infty} z^{-n} x(z) dz = \frac{-1}{2\pi i} \oint_{z=0} z^{n-2} x(z^{-1}) dz. \quad (10)$$

An evaluation module $V(\zeta)$ is a special case of a localized module, on which the positive modes $x[n]_{\zeta}$ with $n > 0$ and $x \in \mathfrak{g}$ act trivially.

Let V be a cyclic $\mathfrak{g}[t]$ -module with cyclic vector v . Then V is endowed with a \mathfrak{g} -equivariant grading inherited from the grading of $U := U(\mathfrak{g}[t])$. The filtered components of V are $\mathcal{F}(n) = U^{\leq n} v$, where $U^{\leq n}$ is the subspace of homogeneous degree in t bounded by n . The grading on V is the associated graded space of this filtration, $\bigoplus_{n \geq 0} \mathcal{F}(n) / \mathcal{F}(n-1)$.

As the filtration is \mathfrak{g} -equivariant, the graded components are \mathfrak{g} -modules.

2.2. Chari's KR -modules of $\mathfrak{g}[t]$

A special case of the construction described in the previous subsection is given as follows. Consider $\mathfrak{g}[t]$ -modules with a highest weight $m\omega_{\alpha}$ ($m \geq 0$ and ω_{α} a fundamental \mathfrak{g} -weight) defined as the cyclic module generated by a highest weight vector v , with relations

$$\begin{aligned} x[n]_{\zeta} v &= 0 \quad \text{if } x \in \mathfrak{n}_+ \text{ and } n \geq 0; \\ h_{\beta}[n]_{\zeta} v &= \delta_{n,0} \delta_{\alpha,\beta} m v; \\ f_{\beta}[n]_{\zeta} v &= 0 \quad \text{if } n \geq \delta_{\alpha,\beta}; \\ f_{\alpha}[0]_{\zeta}^{m+1} v &= 0. \end{aligned} \quad (11)$$

$$(12)$$

We refer to these modules as $C_{\alpha,m}(\zeta)$.¹¹ Their graded version has been previously considered by Chari and Moura.²⁰ The graded components of the associated graded space corresponding to the filtration by homogeneous degree are, of course, \mathfrak{g} -modules.

Except in the case of $\mathfrak{g} = A_r$, these modules are not necessarily irreducible under restriction to the action of \mathfrak{g} . However, $C_{\alpha,m}(\zeta)$ does have a highest weight component isomorphic to the \mathfrak{g} -module $V(m\omega_{\alpha})$, which

appears with multiplicity 1, all other components having a smaller highest weights in the total ordering.

It has not been directly proven (except in special cases) that these modules have the same \mathfrak{g} -decomposition as the Yangian KR-modules, but this theorem will follow from the proof of the Feigin-Loktev conjecture below.

2.3. Fusion products and the Feigin-Loktev conjecture

Consider a set of cyclic $\mathfrak{g}[t]$ -modules $\{V_1(\zeta_1), \dots, V_N(\zeta_N)\}$ localized at pairwise distinct points in \mathbb{C} , $\{\zeta_1, \dots, \zeta_N\}$. Denote the chosen cyclic vector of $V_i(\zeta_i)$ by v_i . If $V_i(\zeta_i)$ are finite-dimensional, so is the space $U(\mathfrak{g}[t])v_1 \otimes \dots \otimes v_N$. Moreover, it has a finite filtration by homogeneous degree in t . The Feigin-Loktev fusion product⁶ is the associated graded space of this filtration. We denote the fusion product by $\mathcal{F}_{\mathbf{V}}^*$. As the grading of $\mathcal{F}_{\mathbf{V}}^*$ is \mathfrak{g} -equivariant, the graded components are \mathfrak{g} -modules. The graded multiplicity of the irreducible \mathfrak{g} -module $V(\lambda)$ in the fusion product is a certain polynomial generating function for the multiplicities in the graded components.

The Feigin-Loktev conjecture is that this polynomial is independent of the localization parameters ζ_i for sufficiently well-behaved \mathfrak{g} -modules, and that in the case that V_i are KR-modules, the graded multiplicity of $V(\lambda)$ is equal to the M -sum formula (2). The equality was proven for \mathfrak{sl}_2 -modules in⁵ and for symmetric power representations of \mathfrak{sl}_n in.²¹

In this paper, we consider only fusion products of KR-modules. They are generated by the highest weight vector v localized at ζ and the relations are those in (12). Let $\mathbf{V} = (V_1, \dots, V_N)$ be a collection of KR-modules of $\mathfrak{g}[t]$ localized at distinct complex numbers $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_N)$.

We parametrize the collection \mathbf{V} by their highest weights $\mathbf{n} = (n_j^{(\alpha)} : 1 \leq \alpha \leq r, j \geq 0)$, meaning that in \mathbf{V} there are exactly $n_j^{(\alpha)}$ KR-modules with highest weight $j\omega_\alpha$. We call this fusion product $\mathcal{F}_{\mathbf{n}}^*$.

3. Functional realization of fusion spaces

We make use of the fact that $\mathfrak{g}[t] \subset \widehat{\mathfrak{g}}$, therefore given a $\mathfrak{g}[t]$ -module V , we can consider the integrable modules induced at some fixed integer level k . We choose k to be sufficiently large, so that the tensor products we consider below have a decomposition determined by the Littelwood-Richardson rule rather than the Verlinde rule. We choose integrable $\widehat{\mathfrak{g}}$ -modules since they have the property that they are completely reducible. Note that smaller values of k are also of interest, and computing the graded fusion product at finite k is still an open problem for the most part.

Consider the action of products of (generating functions of) elements of the affine algebra $\widehat{\mathfrak{g}}$ on the tensor product of highest weight vectors v_i of KR-modules localized at distinct points ζ_i . We use the generating functions $f_\alpha(t) := \sum_{n \in \mathbb{Z}} f_\alpha[n] t^{-n-1}$, where f_α is the element of \mathfrak{n}_- corresponding to the simple root α . We define $\mathcal{F}_{\lambda, \mathbf{n}}^* = \text{Hom}(\mathcal{F}_{\mathbf{n}}^*, V(\lambda))$, where \mathbf{n} parametrizes the set of KR-modules in the fusion product.

The dual space of $\mathcal{F}_{\lambda, \mathbf{n}}^*$ is the associated graded space of $\mathcal{C}_{\lambda, \mathbf{n}}$, consisting of all correlation functions the form

$$\langle u_\lambda | f_{\alpha_1}(t_1) \cdots f_{\alpha_M}(t_M) | v_1 \otimes \cdots \otimes v_N \rangle \quad (13)$$

Here, $M \geq 0$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_M) \in I_r^M$ where $I_r = [1, \dots, r]$. The action of the currents is the fusion action of the previous section. The vector u_λ is the lowest weight vector of the module V localized at $\zeta = \infty$, dual to the highest weight module localized at 0 with \mathfrak{g} -highest weight λ .

This space has a filtration by the homogeneous degree in t_j , and its associated graded space is the graded multiplicity space of the \mathfrak{g} -module $V(\lambda)$ in the fusion product $\mathcal{F}_{\mathbf{n}}^*$.

3.1. Characterization of functions in $\mathcal{C}_{\lambda, \mathbf{n}}$

We now fix λ and \mathbf{n} , and characterize functions in the space $\mathcal{C}_{\lambda, \mathbf{n}}$ according to their symmetry, pole and zero structure:²

- (1) **Zero weight condition:** The correlation function (13) is \mathfrak{g} -invariant. Therefore it must have total \mathfrak{g} -weight equal to 0, which means that

$$0 = \ell^{(\alpha)} + \sum_{\beta, j} C_{\alpha, \beta} m_j^{(\beta)} - \sum_j j n_j^{(\alpha)}, \quad 1 \leq \alpha \leq r, \quad (14)$$

where $\lambda = \sum_\alpha \ell^{(\alpha)} \omega_\alpha$. This fixes $\{m^{(1)}, \dots, m^{(r)}\}$.

For convenience, we rename the variables to keep track of the root α of the generating function in which they appear. Thus, we have functions in the variables $\{t_1, \dots, t_M\} = \{t_i^{(\alpha)} : \alpha \in I_r, 1 \leq i \leq m^{(\alpha)}\}$, where $m^{(\alpha)}$ is the number of generators with root α . Note that the space $\mathcal{C}_{\lambda, \mathbf{n}}$ is the direct sum of spaces of the with fixed $\mathbf{m} = (m^{(1)}, \dots, m^{(r)})$.

- (2) **Pole structure:** Functions in $\mathcal{C}_{\lambda, \mathbf{n}}$ have at most a simple pole when $t_i^{(\alpha)} = t_j^{(\beta)}$ if $C_{\alpha, \beta} < 0$. This is due to the relations in the algebra, which, in the language of generating functions, means that $f_\alpha(t) f_\beta(u) \sim \frac{f_{\alpha+\beta}(t)}{t-u} + \text{non-singular terms}$. We are therefore led to define the less singular function $g(t)$ for each $f(t) \in \mathcal{C}_{\lambda, \mathbf{n}}$:

$$g(\mathbf{t}) := \prod_{\alpha < \beta, C_{\alpha, \beta} < 0} \prod_{i, j} (t_i^{(\alpha)} - t_j^{(\beta)}) f(\mathbf{t}), \quad f(\mathbf{t}) \in \mathcal{C}_{\lambda, \mathbf{n}}. \quad (15)$$

- (3) **Symmetry:** The function $g(\mathbf{t})$ is symmetric under the exchange $t_i^{(\alpha)} \leftrightarrow t_j^{(\alpha)}$. This is due to the fact that $[f_\alpha(t_1), f_\alpha(t_2)] = 0$.
- (4) **Serre condition:** Let α, β be simple roots such that $C_{\alpha, \beta} < 0$ and define $m_{\alpha, \beta} = 1 - C_{\alpha, \beta}$. Then there is a Serre relation in \mathfrak{g} (hence a corresponding relation in $\widehat{\mathfrak{g}}$) of the form $\text{ad}(f_\alpha)^{m_{\alpha, \beta}} f_\beta = 0$. In generating function language,

$$f_\alpha(t_1^{(\alpha)}) \cdots f_\alpha(t_{m_{\alpha, \beta}}^{(\alpha)}) t_\beta(t_1^{(\beta)})$$

has no singularity when all the variables are set equal to each other. This implies that the function $g(\mathbf{t})$ of (15) has the following vanishing property:

$$g(\mathbf{t}) \Big|_{t_1^{(\alpha)} = \cdots = t_{m_{\alpha, \beta}}^{(\alpha)} = t_1^{(\beta)}} = 0.$$

This cancels out the pole which would otherwise appear in the function $f(\mathbf{t})$.

- (5) **Degree restriction:** As u_λ is a lowest weight vector of the module localized at infinity, positive currents $f_\alpha[n]_\infty$ with $n \geq 0$ act on it trivially. The action is given by taking the contour integral at infinity (see Equation (10)), or equivalently, a residue taken at 0. That is, there should be no residue when integrating $t^{n-2} f(t^{-1})$, with $n \geq 0$, at $t = 0$. This gives a degree restriction on the function $f(\mathbf{t}) \in \mathcal{C}_{\lambda, \mathbf{n}}$ for each of the variables:

$$\deg_{t_i^{(\alpha)}} f(\mathbf{t}) \leq -2.$$

- (6) **Poles at ζ_i :** The relation (11) implies that $f(\mathbf{t}) \in \mathcal{C}_{\lambda, \mathbf{n}}$ may have a simple pole at $t_i^{(\alpha)} = \zeta_j$ only if the highest weight of the module localized at ζ_j is a multiple of ω_α , in accordance with the Equation (9). Otherwise, $f[n]_{\zeta_i} v(\zeta_i) = 0$ if $n \geq 0$. Moreover, we have
- (7) **Integrability condition:** We assume each module V_k has highest weight $\ell_k \omega_{\alpha_k}$. The relation (12) requires that $g_2(\mathbf{t}) = \left(\prod_{\alpha, j, k} (t_j^{(\alpha)} - \zeta_k)^{\delta_{\alpha, \alpha_k}} \right) g(\mathbf{t})$ has the following vanishing property:

$$g_2(\mathbf{t}) \Big|_{t_1^{(\alpha)} = \cdots = t_{\ell_k+1}^{(\alpha)} = \zeta_k} = 0.$$

These conditions characterize the space $\mathcal{C}_{\lambda, \mathbf{n}}$ completely. The only difficulty is to compute its Hilbert polynomial. This is done by introducing another filtration on the space of functions. The idea for such a filtration was first introduced by Feigin and Stoyanovsky.²²

3.2. Filtration of the space of functions

Let $\lambda = (\lambda^{(1)}, \dots, \lambda^{(r)})$ where $\lambda^{(\alpha)}$ is a partition of $m^{(\alpha)}$. Let $m_a^{(\alpha)}$ denote the number of parts of $\lambda^{(\alpha)}$ equal to a . Thus, $\sum_a a m_a^{(\alpha)} = m^{(\alpha)}$. Fix a standard tableau for each partition (the result is independent of the choice of tableaux, and when we discuss a partition below we always refer to the fixed tableau) and define the evaluation map $\text{ev}_\lambda : \mathcal{C}_{\lambda, \mathbf{n}}[m^{(1)}, \dots, m^{(r)}] \rightarrow \mathcal{H}[\lambda]$, where $\mathcal{H}[\lambda]$ is the space of functions in several variables: one variable for each row of each partition in λ .

The evaluation map is defined as follows. If the letter i appears in the j th row of length a in $\lambda^{(\alpha)}$, then $\text{ev}_\lambda(t_i^{(\alpha)}) = u_{a,j}^{(\alpha)}$. This is extended by linearity to $\mathcal{C}_{\lambda, \mathbf{n}}$.

We order multipartitions lexicographically, and define

$$\Gamma_\lambda = \bigcap_{\mu > \lambda} \ker \text{ev}_\mu.$$

This gives a finite filtration of $\mathcal{C}_{\lambda, \mathbf{n}}$, with $\Gamma_\mu \subset \Gamma_\lambda$ if $\mu < \lambda$. We consider the image of the graded components $\Gamma_\lambda / (\Gamma_\lambda \cap \ker \text{ev}_\lambda)$ under the evaluation map ev_λ .

Again, this is a space of functions, isomorphic to a subspace of $\mathcal{H}[\lambda]$. Let us denote its image by $\tilde{\mathcal{H}}[\lambda]$. Its characterization is as follows.

- (1) **Symmetry:** Functions in Γ_λ are symmetric in the variables $\{t_1^{(\alpha)}, \dots, t_{m^{(\alpha)}}^{(\alpha)}\}$ for each α . The full symmetry is lost under the evaluation map, but the functions are still symmetric with respect to the variables labeled by rows of the same length in $\lambda^{(\alpha)}$. That is, they are symmetric with respect to the exchange of variables $\{u_{a,1}^{(\alpha)}, \dots, u_{a,m_a^{(\alpha)}}^{(\alpha)}\}$ for each a, α .
- (2) Functions in Γ_λ are in the kernel of any evaluation ev_μ with $\mu > \lambda$, which means that functions in the image vanish whenever we set the variables corresponding to different rows of the same partition equal to each other. In fact, one can prove that

Lemma 3.1. *Functions in $\tilde{\mathcal{H}}[\lambda]$ have a factor $(u_{a,j}^{(\alpha)} - u_{a,k}^{(\alpha)})^{2 \min(a,b)}$ for all $j < k$.*

- (3) **Pole structure and Serre condition:** The pole at $t_i^{(\alpha)} = t_j^{(\beta)}$ when $C_{\alpha,\beta} < 0$, together with the vanishing condition of $g(\mathbf{t})$ which follows from the Serre relation, implies that functions in $\tilde{\mathcal{H}}[\lambda]$ have a pole of order at most $\min(|C_{\alpha,\beta}|b, |C_{\alpha,\beta}|a)$ whenever $u_{a,i}^{(\alpha)} = u_{b,j}^{(\beta)}$ (inherited from conditions (1) and (3) of the previous subsection).

- (4) **Poles at ζ_j :** The pole at $t_i^{(\alpha)} = \zeta_j$ in case V_j has highest weight proportional to ω_α , together with the integrability of that module, translate to the following statement for $f \in \tilde{\mathcal{H}}[\lambda]$: There is a pole of order at most $\min(\ell, a)$ at $u_{a,i}^{(\alpha)} = \zeta_j$ if V_j has a highest weight equal to $\ell_j \omega_\alpha$. We define $\delta(j, \alpha) = 1$ if the highest weight of V_j is a multiple of ω_α , and $\delta(j, \alpha) = 0$ otherwise.
- (5) **Degree restriction:** Functions in $\tilde{\mathcal{H}}[\lambda]$ have a degree in $u_{a,j}^{(\alpha)}$ which is bounded from above by $-2a$.

We do not know that these are all the conditions on functions in $\tilde{\mathcal{H}}[\lambda]$: The map ev_λ is injective by definition but not necessarily surjective. However, we can compute the Hilbert polynomial of the space \mathcal{F} defined by the conditions above, which gives an upper bound on the Hilbert polynomial of $\tilde{\mathcal{H}}[\lambda]$.

To summarize, we know that $f(\mathbf{u}) \in \mathcal{F}$ has the form

$$\frac{\prod_{(a,i) \neq (b,j)} (u_{a,i}^{(\alpha)} - u_{b,j}^{(\alpha)})^{\min(a,b)} \times f^0(\mathbf{u})}{\prod_{a,i,j} (u_{a,i}^{(\alpha)} - \zeta_j)^{\delta(j,\alpha) \min(\ell_j, a)} \prod_{a,i,b,j} \prod_{\alpha < \beta: C_{\alpha,\beta} < 0} (u_{a,i}^{(\alpha)} - u_{b,j}^{(\alpha)})^{\min(|C_{\alpha,\beta}|b, |C_{\beta,\alpha}|a)}},$$

where $f^0(\mathbf{u})$ is a polynomial in \mathbf{u} , symmetric under the exchange $u_{a,i}^{(\alpha)} \leftrightarrow u_{a,j}^{(\alpha)}$, of degree such that

$$\deg_{u_{a,j}^{(\alpha)}} f(\mathbf{u}) \leq -2a.$$

To compute the Hilbert polynomial we set all $\zeta_j = 0$ so that the function above is homogeneous in \mathbf{u} . That is, we compute the Hilbert polynomial of the associated graded space. It is very important to note that the values of ζ_j do not affect the value of the Hilbert polynomial, that is, there is no change in the q -dimensions of the space when we take the associated graded space.

The degree in $u_{a,j}^{(\alpha)}$ of the prefactor of $f^0(\mathbf{u})$ is $-2a - P_a^{(\alpha)}$, where $P_a^{(\alpha)}$ is defined in Equation (4). Moreover, the overall homogeneous degree of the prefactor is $Q(\mathbf{m}, \mathbf{n})$ as defined in equation (5). The Hilbert polynomial of the space of symmetric functions in m variables of degree less than or equal

to p is the q -binomial coefficient,

$$\left[\begin{matrix} m+p \\ m \end{matrix} \right]_q = \frac{\prod_{i=1}^{m+p} (1 - q^i)}{\prod_{i=1}^m (1 - q^i) \prod_{i=1}^p (1 - q^i)}.$$

Therefore, the Hilbert polynomial of \mathcal{F} is

$$q^{Q(\mathbf{m}, \mathbf{n})} \prod_{\alpha, j} \left[\begin{matrix} m_j^{(\alpha)} + p_j^{(\alpha)} \\ m_j^{(\alpha)} \end{matrix} \right]_q,$$

which is the upper bound (at each degree in q) of the Hilbert polynomial of $\tilde{\mathcal{H}}[\lambda]$, since it is a polynomial with positive coefficients.

Summing over the graded components, there follows the main Theorem:

Theorem 3.1.² *The Hilbert polynomial of $\mathcal{C}_{\lambda, \mathbf{n}}$, which is the Feigin-Loktev fusion product of KR -modules, is bounded from above by $M_{\lambda, \mathbf{n}}(q)$ defined in Equation (2).*

4. Proof of the $M = N$ conjecture

In this section, we explain the proof³ of the identity (7). For ease of readability, we explain the technique explicitly for the Lie algebra \mathfrak{sl}_2 , and then state the key ingredients necessary in the generalization to arbitrary Lie algebras. The only difficulty in this generalization is the rapid proliferation of indices.

4.1. The case of \mathfrak{sl}_2

As explained in the introduction, one need only prove the $M = N$ identity only for the case $q = 1$ for the pentagon of identities to hold, due to the positivity of the M -sum. In the case of \mathfrak{sl}_2 , we drop the root superscript (α) in the vacancy numbers $P_i^{(\alpha)}$ and so forth.

Fix $\mathbf{n} = (n_1, \dots, n_k) \in \mathbb{Z}_+^k$ and an \mathfrak{sl}_2 -highest weight $\ell\omega_1$ with $\ell \in \mathbb{Z}_+$. Consider the following generating function:

$$Z_{\ell, \mathbf{n}}^{(k)}(x_0, x_1) = \sum_{\mathbf{m} \in \mathbb{N}^k} x_1^{-q_0} x_0^{q_1} \prod_{i=1}^k \binom{m_i + q_i}{m_i} \quad (16)$$

Here, we have defined

$$q_i = \ell + \sum_{j=i+1}^k (j-i)(2m_j - n_j), \quad i \geq 0.$$

In particular, notice that when $q_0 = 0$, $q_i = P_i$ for all $i > 0$.

The binomial coefficient is defined as usual

$$\binom{m+p}{m} = \frac{(m+p)!}{m!p!}.$$

This is well-defined for both negative and positive values of p , and when $p < 0$ it has an overall sign -1^m .

This N -sum can be obtained from this generating function as follows. First, here and below, we note that in the N and M -sums, $m_j = 0$ if $j > k$ in (2). However all the identities we prove are valid under this restriction; since only a finite number of the m_j make a non-trivial contribution to the summation (2), one can take $k \rightarrow \infty$ at the end of the day with no loss of generality.

Second, in both the N and M sums, there is a “weight restriction” restriction on the \mathbf{m} -summation. This is equivalent to setting $q = 0$, or alternatively, considering only the constant term in x_1 in the generating function. expression. We do not restrict the sum to $P_i \geq 0$ yet, but in the M and N sums, the variable x_0 must be set to 1.

Lemma 4.1. *There is a recursion relation,*

$$Z_{\ell, (n_1, \dots, n_k)}^{(k)}(x_0, x_1) = \frac{x_1^{n_1+2}}{x_0 x_2} Z_{\ell, (n_2, \dots, n_k)}^{(k-1)}(x_1, x_2),$$

where x_i are solutions of the A_1 Q -system or cluster algebra mutation²³ with arbitrary boundary conditions:

$$x_{i+1}x_{i-1} = x_i^2 - 1, \quad i \in \mathbb{Z}.$$

Proof. The variable m_1 is not part of the expression for q_1 so we can perform the summation over m_1 , using the identity

$$\sum_{m_1 \geq 0} x_1^{-2m_1} \binom{m_1 + q_1}{m_1} = \left(\frac{x_1^2}{x_1^2 - 1} \right)^{q_1+1} = \left(\frac{x_1^2}{x_0 x_2} \right)^{q_1+1},$$

where we have used the Q -system in the second equality.

We separate out the dependence on m_1 in the summand, and note that

$$2q_i - q_{i-1} + 2m_i - n_i = q_{i+1}.$$

Moreover, if we denote by $q_i^{(j)}$ the function q_i with arguments being of the last $j - i$ variables in the list (m_1, \dots, m_k) (so that $q_i^{(k)} = q_i$), then $q_i^{(k-1)} = q_{i+1}^{(k)}$, or $q_{i-1}^{(k-1)} = q_i^{(k)}$.

We have

$$\begin{aligned}
Z_{\ell, (n_1, \dots, n_k)}^{(k)}(x_0, x_1) &= \sum_{m_1, \dots, m_k} x_1^{-q_0} x_0^{q_1} \prod_{j=1}^k \binom{m_j + q_j}{m_j} \\
&= \sum_{m_2, \dots, m_k} x_0^{q_1} \prod_{j \geq 2} \binom{m_j + q_j}{m_j} x_1^{n_1 + q_2 - 2q_1} \sum_{m_1} x_1^{-2m_1} \binom{m_1 + q_1}{m_1} \\
&= \sum_{m_2, \dots, m_k} x_0^{-1} x_1^{n_1 + q_2 + 2} x_2^{-1 - q_1} \prod_{j=2}^k \binom{m_j + q_{j-1}^{(k-1)}}{m_j} \\
&= \frac{x_1^{n_1 + 2}}{x_0 x_2} \sum_{m_2, \dots, m_k} x_2^{-q_0^{(k-1)}} x_1^{q_1^{(k-1)}} \prod_{j=1}^{k-1} \binom{m_{j+1} + q_j^{(k-1)}}{m_{j+1}} \\
&= \frac{x_1^{n_1 + 2}}{x_0 x_2} Z_{\ell, (n_2, \dots, n_k)}^{(k-1)}(x_1, x_2).
\end{aligned}$$

(Here, the superscript $(k-1)$ on q_0, q_1 means we take these variables as defined for the $k-1$ variables with indices $2, \dots, k$.) \square

Using the Lemma, by induction, we see that the generating function factorizes:

$$Z_{\ell, \mathbf{n}}^{(k)}(x_0, x_1) = \frac{x_1 x_k^{\ell+1}}{x_0 x_{k+1}^{\ell+1}} \prod_{i=1}^k x_i^{n_i}. \quad (17)$$

In particular,

$$Z_{\ell, \mathbf{n}}^{(k)}(x_0, x_1) = Z_{0, (n_1, \dots, n_p)}^{(p-1)}(x_0, x_1) Z_{\ell, (n_{p+1}, \dots, n_k)}^{(k-p+1)}(x_{p-1}, x_p). \quad (18)$$

We are interested in the constant term in x_1 in $Z_{\ell, \mathbf{n}}^{(k)}(x_0, x_1)$. We use the factorization Lemma for the first factor, and the definition via summation for the second factor:

$$Z_{\ell, \mathbf{n}}^{(k)}(x_0, x_1) = \frac{x_1 x_{p-1}}{x_0 x_p} \prod_{j=1}^{p-1} x_j^{n_j} \sum_{m_p, \dots, m_k} x_p^{-q_{p-1}} x_p^{q_p} \prod_{j=p}^k \binom{m_j + q_j}{m_j}. \quad (19)$$

Suppose we restrict the summation in the second factor to $q_p \geq 0$ only. Moreover, we are interested in the generating function when $x_0 = 1$. In this case, all x_i are polynomials in x_1 (Chebyshev polynomials of the second type). Terms in the summation in which $q_{p-1} < 0$ are therefore products of polynomials in x_1 since there are no factors of x_i in the denominator in this case. Moreover, there is an overall factor of x_1 , so that there is no constant term in x_1 in this case. Thus,

Lemma 4.2. *If the summation over (m_p, \dots, m_k) in (19), is restricted to $q_p \geq 0$, then only terms with $q_{p-1} \geq 0$ contribute to the constant term in x_1 when $x_0 = 1$.*

We use an induction argument, where the base step is clear ($q_k = \ell$), to conclude that the only terms which contribute to the constant term in x_1 are terms from the restricted summation, $q_i \geq 0$ ($i > 0$). When $q_0 = 0$, this is the $N = M$ identity, since $q_i = P_i$ in that case.

4.2. The simply-laced case

This case is a straightforward generalization of the \mathfrak{sl}_2 case^a.

We now define the generating function

$$Z_{\lambda, \mathbf{n}}^{(k)}(\mathbf{x}_0, \mathbf{x}_1) = \sum_{\mathbf{m}} \mathbf{x}_0^{-\mathbf{q}_1} \mathbf{x}_1^{\mathbf{q}_0} \prod_{\alpha, j} \binom{m_j^{(\alpha)} + q_j^{(\alpha)}}{m_j^{(\alpha)}},$$

(as is the norm, when \mathbf{x} and \mathbf{q} represent vectors indexed by the same set, we write $\mathbf{x}^{\mathbf{q}}$ for the product over the components.) Here, $\lambda = \sum_{\alpha=1}^r \ell^{(\alpha)} \omega_{\alpha}$, $\mathbf{n} = (n_j^{(\alpha)})_{\alpha \in I_r, j \in I_k}$, the summation is over $\mathbf{m} = \{m_j^{(\alpha)}, \alpha \in I_r, j \in I_k\}$ non-negative integers, and we define

$$q_i^{(\alpha)} = \ell^{(\alpha)} + \sum_{j=i+1}^k \sum_{\beta \in I_r} (j-i)(C_{\alpha, \beta} m_j^{(\alpha)} - \delta_{\alpha, \beta} n_j^{(\alpha)}).$$

When $q_{\alpha, 0} = 0$ for all α , this corresponds to the “weight restriction” (3) in the M and N -sums, and in that case, $q_i^{(\alpha)} = P_i^{(\alpha)}$ if $i > 0$. We have now $2r$ variables $\mathbf{x}_0 = (x_{1,0}, \dots, x_{r,0})$ and $\mathbf{x}_1 = (x_{1,1}, \dots, x_{r,1})$. The generating function is related to the M or N -sums when we evaluate the sum at $x_{\alpha,0} = 1$ and consider the constant term in \mathbf{x}_1 .

Following the steps outlined for \mathfrak{sl}_2 we derive a recursion relation for the generating function:

$$Z_{\lambda, \mathbf{n}}^{(k)}(\mathbf{x}_0, \mathbf{x}_1) = \frac{\mathbf{x}_1^{2+\mathbf{n}_1}}{\mathbf{x}_0 \mathbf{x}_2} Z_{\lambda, \mathbf{n}^{(k-1)}}^{(k-1)}(\mathbf{x}_1, \mathbf{x}_2)$$

where $\mathbf{n}^{(k-1)}$ is \mathbf{n} with $\mathbf{n}_1 = \mathbf{0}$. Here, we have defined $x_{\alpha, i}$ to be the solutions of the following system:

$$x_{\alpha, i+1} x_{\alpha, i-1} = x_{\alpha, i}^2 - \prod_{C_{\alpha, \beta} = -1} x_{\beta, i}.$$

^aBelow, we have two sets of indices for \mathbf{x} , \mathbf{n} etc. When we write \mathbf{x}_0 we mean the collection of r elements $(x_{1,0}, \dots, x_{r,0})$, and so forth.

This is called the Q -system for the simply-laced Lie algebra \mathfrak{g} , provided we set the initial conditions $x_{\alpha,0} = 1$. Otherwise it is a cluster algebra mutation,²³ and therefore, under these special initial conditions, all its solutions are polynomials in the variables $x_{\beta,1}$.²⁴

We again repeat the arguments of the previous section to factorize the generating function:

$$Z_{\lambda,\mathbf{n}}^{(k)}(\mathbf{x}_0, \mathbf{x}_1) = \prod_{\alpha} \frac{x_{\alpha,1} x_{\alpha,k}^{\ell_{\alpha}+1}}{x_{\alpha,0} x_{\alpha,k+1}^{\ell_{\alpha}+1}} \prod_{j=1}^k x_{\alpha,j}^{n_j^{(\alpha)}},$$

from which we deduce that

$$Z_{\lambda,\mathbf{n}}^{(k)}(\mathbf{x}_0, \mathbf{x}_1) = \frac{\mathbf{x}_1 \mathbf{x}_{p-1}}{\mathbf{x}_0 \mathbf{x}_p} \prod_{j=1}^{p-1} \mathbf{x}_j^{n_j} \sum_{\mathbf{m}^{(p)}} \mathbf{x}_p^{-\mathbf{q}_{p-1}} \mathbf{x}_{p-1}^{\mathbf{q}_p} \prod_{j=p}^k \binom{\mathbf{m}_j + \mathbf{q}_j}{\mathbf{q}_j}.$$

Here, $\mathbf{m}^{(p)}$ are the last $k-p+1$ components of the list $(\mathbf{m}_1, \dots, \mathbf{m}_k)$. A binomial coefficient with vector entries is notation for the product of binomial coefficients over the components.

Suppose we restrict the summation to $\mathbf{m}^{(p)}$ such that $q_p^{(\alpha)} \geq 0$ for some α , and such that $q_{p-1}^{(\alpha)} < 0$ for the same α . We look for a contribution to the constant term in $x_{\alpha,1}$. All \mathbf{x}_i are polynomials in \mathbf{x}_1 after evaluation at $x_{\alpha,0} = 1$ for all α . Terms with $q_p^{(\alpha)} < 0$ do not have a factor $x_{\alpha,p}$ in the denominator, and are therefore polynomials in $x_{\alpha,i}$ for several i and fixed α . One can show that $\prod_{\beta \neq \alpha} x_{\beta,p}^{-1}$ has no negative powers of $x_{\alpha,1}$ (see,³ Lemma 4.8). Therefore we have a polynomial in $x_{\alpha,1}$, with an overall power of $x_{\alpha,1}$, hence there is no constant term in $x_{\alpha,1}$. We repeat this argument for each α and inductively for each p starting from $p = k$, until we get

Lemma 4.3. *There is no contribution to the constant term in \mathbf{x}_1 in the summation from terms with $q_j^{(\alpha)} < 0$ for any p, j , hence from terms with $P_j^{(\alpha)} < 0$ when we consider the terms with $q_0^{(\alpha)} = 0$.*

This implies that $M = N$ for the simply-laced Lie algebras.

4.3. The non-simply laced case

This case is less elegantly derived, as it requires the introduction of even more variables in the generating function, and each case must be treated separately. Nevertheless, the argument goes through in the same (more involved) manner. In the process we must define the set of variables which

satisfy the following system of equations:

$$x_{\alpha, i+1} x_{\alpha, i-1} = x_{\alpha, i} - \prod_{\beta: C_{\alpha, \beta} < 0} \prod_{j=0}^{-C_{\alpha, \beta}-1} x_{\beta, [(C_{\beta, \alpha}|i+j)/|C_{\alpha, \beta}|]}.$$

If $x_{\alpha, 0} = 1$ for all α , then the equation for $i > 0$ is known as the Q -system (for the simple Lie algebra with Cartan matrix C), and it is known to be satisfied by the characters $x_{\alpha, i}$ of the KR-modules $\text{KR}_{\alpha, i}$ if $x_{\alpha, 1}$ is the character of the fundamental module.

We find in the $k \rightarrow \infty$ limit that

Theorem 4.1.³ *For any simple Lie algebra and λ a dominant weight, \mathbf{n} a vector in $\mathbb{Z}_+^{r \times k}$, $M_{\lambda, \mathbf{n}} = N_{\lambda, \mathbf{n}}$.*

5. Summary

Prior to the work described in the previous section, it was known that for any simple Lie algebra, the multiplicity of the $U_q(\mathfrak{g})$ -module with \mathfrak{g} highest weight λ in the tensor product of Kirillov-Reshetikhin modules is the N -sum formula. This followed theorems of Hatayama et al⁸ and Nakajima's theorem about the q -characters for T -systems corresponding to simply-laced Lie algebras,⁹ as well as the extension by Hernandez for other algebras.¹⁰

We now have all the equalities in the pentagon of identities. That is, since we have proven that $M = N$, we have proven also the following:

Corollary 5.1. *The multiplicity of the \mathfrak{g} -module $V(\lambda)$ in the tensor product of Chari's KR modules of $\mathfrak{g}[t]$ is equal to the multiplicity of the $U_q(\mathfrak{g})$ module with \mathfrak{g} -highest weight λ in the corresponding tensor product of $U_q(\widehat{\mathfrak{g}})$ of Kirillov-Reshetikhin type.*

Corollary 5.2. *The Hilbert polynomial of the graded multiplicity space of $V(\lambda)$ in the Feigin-Loktev fusion product is the fermionic M -sum (generalized Kostka polynomials in the case of A_n). This is the Feigin-Loktev conjecture.*

Corollary 5.3. *The Bethe integer sets (parametrizing Bethe vectors) in the generalized Heisenberg model as solved by Kirillov and Reshetikhin are in bijection with vectors in the Hilbert space of the model, and therefore the completeness conjecture holds.*

We should remark that although it is well known that not all Bethe states come from the so-called “string hypothesis” in these models, nevertheless this gives a good counting of the states.

The proof described in the previous section shows that the vanishing of the “non-positive” components of the N -sum formula is due to the fact that the solutions of the Q -system with the KR-boundary condition are polynomials in the initial data $x_{\alpha,1}$. This fact is clear, once one refers to the theorem that the solutions $x_{\alpha,n}$ with $n > 0$ are characters of KR-modules, which are in the Grothendieck group generated by $\{x_{1,1}, \dots, x_{r,1}\}$ (the characters of the r fundamental representations). However these facts are not immediately obvious without resorting to the proven theorems on the subject. The cluster algebra formulation of the Q -system gives an entirely combinatorial interpretation for this fact.^{23,24}

The polynomiality property is quite general for a much larger class of cluster algebras, under even more general boundary conditions, which give a certain vanishing of the numerators.²⁴ For example, the same property holds for the T -systems.

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